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NUMERICAL CUMULATIVE DAMAGE: THE FM-MODEL

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Numerical Cumulative Damage: The FM-model

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ABSTRACT

On the basis of the B-model developed in [Bogdanoff, J.L. and F. Kozin; 1985] a new numerical model incorporating the physical knowledge of fatigue crack propagation is developed. The model is based on the Markov assumption and is applicable to deterministic as well as to random loading. Once the model parameters for a given material have been determined, the results can be used for any structure as soon as the geometrical function is known.

KEYWORDS

Random fatigue, fracture mechanics, numerical cumulative damage model, Markov assumption.

1. INTRODUCTION

Varying loads acting on a structure will cause initiation and propagation of cracks. The cumulative damage (CD) is defined as the irreversible accumulation of damage through lifetime, which ultimately causes failure. The process is random and justifies reduction of the reliability of the structure.

A mathematical model which is primarily based on physical observable quantities is necessary to describe the CD-process. Usually, distinction is made between two main groups of damage models: deterministic models and probabilistic models.

Deterministic models only give information about the mean damage accumulation, thus ignoring the fluctuations characteristic for fatigue. The use of a probabilistic model makes it possible to take account of the fluctuations and making the CD-model more realistic. The fluctuations are due to variations in

Initial state: Distribution of the initial damage e.g. initial crack lengths in welds.

Magnitude and order of load cycles resulting in interaction effects in the form of retardation or acceleration of the crack growth rate.

Material properties in the form of e.g. inhomogeneities and loss of isotropy.

all of which influence the crack growth.

From an application point of view, a simple procedure to determine the crack growth progress is desirable.

Firstly, a proper probabilistic model must be chosen. The model developed in this paper is described in chapter 3 and is an applicable model. Secondly, the model parameters must be determined (estimated). The number of model parameters depends on the complexity of the phenomenon considered. Thirdly, typical progresses of the CD-process must be established by simulations. Hereby, the sample functions are determined. A sample function is an event of a random process, in this case it is a function which describes the accumulation of damage in a component as a function of time. Discretizing the time, the damage also has to be discretized, and a measure of damage must be chosen. Finally, the probabilities for given events (e.g. failure at a specified time, numbers of replacements) and the statistical moments (mean value, variance of e.g. lifetime) can be estimated using the sample functions.

Generally, any parameter can be used as a damage measure. The probabilistic, cumulative damage model described in chapter 2 only requires that the damage measure describes a non-decreasing function.

In the model described by [Ditlevsen, O. and R. Olesen; 1985] the crack increment Δa is measured for fixed values of the increment in numbers of load cycles, ΔN, i.e., Δa is random.

The cumulative damage model developed in this paper, see chapter 3, uses Δa as a fixed parameter whereas ΔN to cause Δa is random. This is the most appropriate method because the distribution of numbers of cycles performed to reach a given crack length a, can hereby be determined. Further, as shown in chapter 3, Δa can be regarded as a material constant.

2. THE B-MODEL

In the following a cumulative damage model developed by Bogdanoff and Kozin, see e.g. [Bogdanoff, J.L. and F. Kozin; 1985], [Bogdanoff, J.L.; 1978a], [Bogdanoff, J.L. and W. Krieger; 1978], [Bogdanoff, J.L.; 1978b] and [Bogdanoff, J.L.; 1980], is described. The stochastic model is in this paper referred to as the B-model.

A basic element in the B-model is the division of the load into duty cycles, see figure 2.1. A duty cycle (DC) is defined as a repetitive period of operation in the life of a component during which damage may accumulate.

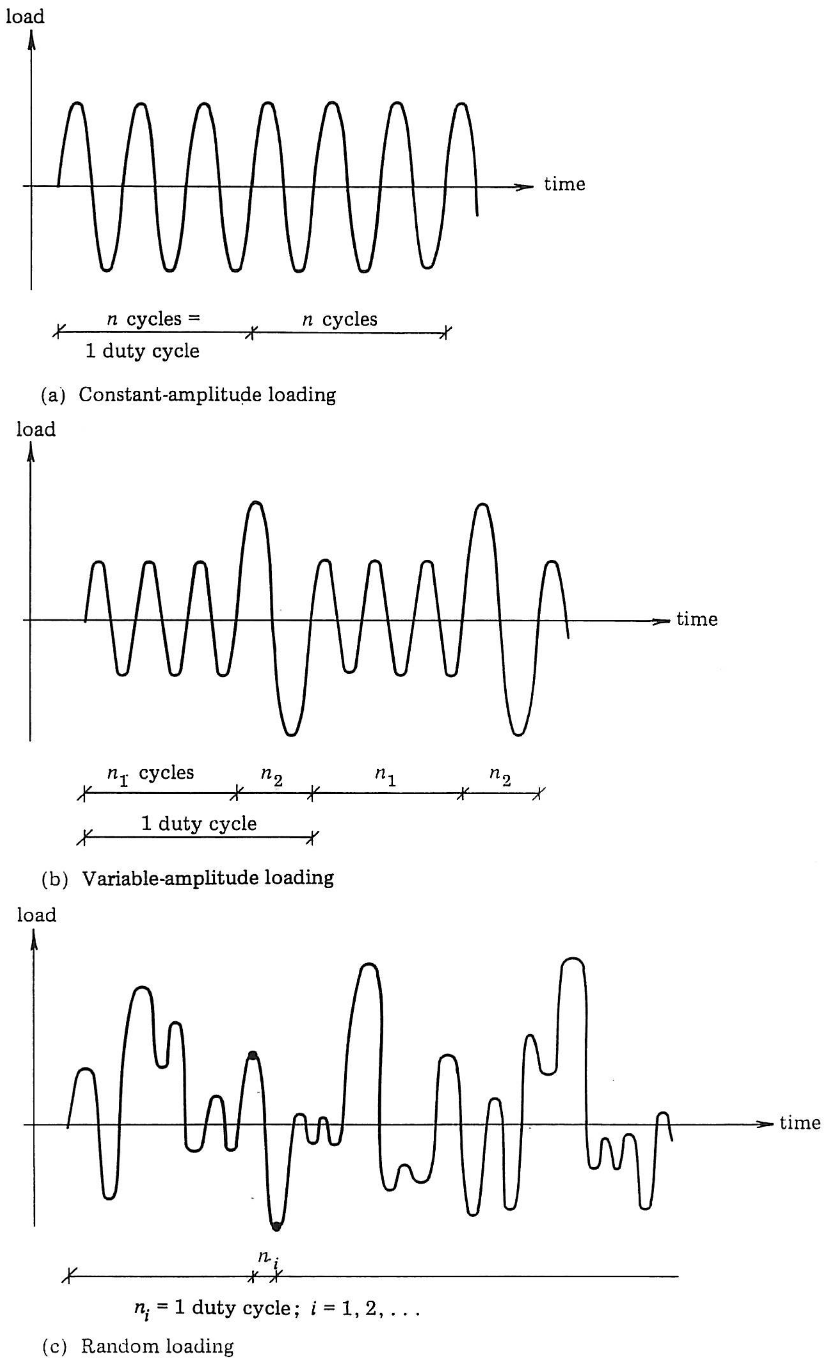

Figure 2.1: Examples of duty cycles.

The discrete time, x, is measured in numbers of DC's, $x = 1, 2, \ldots$. The damage accumulation is considered as a stochastic process in which the possibility of damage accumulation is present each time the structure has experienced a DC.

The damage d is assumed to be discrete with the states $d = 0, 1, 2, ..., b$, where b corresponds to failure. Further, it is assumed that the increment of damage at the end of the DC only depends on the DC itself and the state of damage present at the start of the DC. The damage only increases by one unit at a time.

Hereby, the damage accumulation process can be regarded as a discrete-time, discrete-state Markov process, see e.g. [Bogdanoff, J.L. and Kozin, F; 1985, ch.2] and [Papoulis, A.; 1984, p.385].

Such a Markov process is completely described by its transition matrix (one transition matrix for each duty cycle) and by the initial conditions.

The initial probability distribution of the damage states is given by the vector

$$\overline{p}_0 = \{\pi_0, \pi_1, \pi_2, \ldots, \pi_{b-1}, \pi_b\} \qquad \pi_j \geq 0 \tag{2.1}$$

where

$$\begin{aligned} \pi_j &= \text{prob}\,\{\text{damage is initially in state } j\} \quad j = 0, 1, 2, \ldots, b-1 \\ \pi_b &= 0 \end{aligned}$$

Thus, the π_j-values form the probability mass function of the initial damage state. The distribution of the initial crack lengths in a weld can be given as an example.

As mentioned earlier, the damage only increases by one unit at a time. Thus, it is possible to establish the transition matrix ($b \times b$) for the i th duty cycle $\overline{\overline{P}}_i$ given by

$$\overline{\overline{P}}_i = \begin{bmatrix} p_0 & q_0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_1 & q_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & p_2 & q_2 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p_j & q_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & p_{b-1} & q_{b-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \qquad i = 1, 2, \ldots \tag{2.2}$$

where the conditional probabilities

$$\begin{aligned} p_j &= \text{prob}\,\{\text{remains in state } j |\text{ previously in state } j\} \\ & \qquad j = 0, 1, 2, \ldots, b \\ q_j &= \text{prob}\,\{\text{goes to state } j+1 |\text{ previously in state } j\} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} p_j &\geq 0 \\ q_j &\geq 0 \qquad j = 0, 1, 2, \ldots, b \\ p_j + q_j &= 1 \end{aligned}$$

The damage state at the time x is then given by the vector

$$\overline{p}_x = \overline{p}_0 \; \overline{\overline{P}}_1 \; \overline{\overline{P}}_2 \; \cdots \; \overline{\overline{P}}_x \tag{2.4}$$

where

$\overline{p}_0$ is given by (2.1)
$\overline{p}_x = \{p_x(j)\} = \text{prob}\,\{\text{damage is in state } j \text{ at time } x\}$

If $\overline{\overline{P}}_i$ is identical for all duty cycles, i.e. $\overline{\overline{P}}_i = \overline{\overline{P}}$, (2.4) reduces to

$$\overline{p}_x = \overline{p}_0 \; \overline{\overline{P}}^{\,x} \tag{2.5}$$

The advantage of the B-model is that once the model parameters $\pi_j,\ p_j,\ q_j;\ j = 0, 1, 2, \ldots, b$ are determined, the state of damage in the given structure is available at any time using (2.4). This means that all statistical information about the damage process can be represented by the model.

One of the drawbacks of the model is that a great numbers of parameters are involved and that it is not clear how to determine these parameters. Further, the physical knowledge of the structure is not directly used, but is contained in the model parameters p_j and q_j; $j = 0, 1, 2, \ldots, b$, which are only valid for the given problem.

The B-model has been tested on data originating from several experiments. The most well-known set of data is the Virkler data, cf. [Bogdanoff, J.L and F. Kozin; 1985, ch.4] and [Virkler, D.A., B.M. Hillberry and P.K. Goel; 1979].

The Virkler data were obtained in 68 tests using 2024-T3 aluminium alloy CCT-specimens (Center Cracked Tension). In each test, 164 values of the number of cycles and the crack length, (N, a), were recorded for fixed values of Δa. The estimated mean (N, a)-curve, obtained by using the B-model, has a remarkably smooth progress.

Similar results are found applying the B-model to other data. The good agreement between observed and estimated (N, a)-curves indicate that the B-model, and thus the Markov assumption, is reasonable.

This result is used in chapter 3 where existing knowledge of crack propagation is incorporated into the B-model.

3. THE BASIC IDEAS OF THE FM-MODEL

The purpose of this chapter is to relate the basic ideas of the B-model described in chapter 2 to physical problems in such a way, that it is possible to transfer results from one structure to another. This description is named the Fracture Mechanical Model (FM-model).

As mentioned in chapter 1, the crack length, a, can be used as a damage measure which is advantageous since a is a quantity that can be observed. Further, a reliable connection between the loading process and the process of fatigue, in the form of crack growth, has been established and verified by experiments. The connection is represented by a crack propagation law, which in most cases expresses the crack propagation as a function of the so-called stress intensity factor range ΔK. This is a fracture mechanical value defined by the stress field near a crack tip, c.f. [Hellan, K.; 1985, ch.2].

In the FM-model, the damage is assumed to progress in steps of the length δa. Hereby, δa is assumed to be a material constant. Thus, the state of damage can be defined as

$$a_j = a_0 + j\ \delta a \qquad j = 0, 1, 2, \ldots, b \tag{3.1}$$

where

$$\begin{aligned} a_j &= \text{crack length at damage state } j \text{ [mm]} \\ a_0 &= \text{initial crack length [mm]} \\ a_b &= \text{failure crack length [mm]} \end{aligned}$$

The above has the effect that both the damage process and the crack growth process are stepwise even though a continuous process for the latter would be more realistic from a continuum point of view. Bearing in mind that the material is not a continuum but is inhomogeneous, from a micro point of view, the model seems reasonable.

In a given crack state, $a = a'$, the propagation of the crack is assumed to be described by a Bernoulli random variable X, see e.g. [Benjamin, J.R. and C.A. Cornell; 1970, p.222].

$$X = \begin{cases} 0 & \text{the crack remains in the given state} \\ 1 & \text{the crack propagates } \delta a \end{cases} \tag{3.2}$$

The probability mass function of X then is

$$f_X(x) = \begin{cases} p_j = 1 - q_j & \text{for } X = 0 \\ q_j & \text{for } X = 1 \end{cases} \qquad j = 0, 1, 2, \ldots, b \tag{3.3}$$

The quantity q_j is known as the transition probability, see also (2.3). The crack growth problem is then reduced to the determination of $q_j = q(\Delta K_j) = q(\Delta K(a_j))$.

The most simple situation occurs if ΔK is constant, i.e. the crack tip loading is constant no matter how long the crack is. If so, $\Delta K_j = \Delta K$ and hence, $q_j = q$.

Combining the empirical Paris law, which is one of the most frequently used crack propagation laws, and the binomial distribution, the estimation of q is possible.

Paris law, see [Paris, P.C. and F. Erdogan; 1963], is given as

$$\frac{da}{dN} = C(\Delta K)^m \tag{3.4}$$

where

da	=	increase in crack length [mm]
dN	=	increase in numbers of cycles
C	=	material constant [mm/(MPa$\sqrt{\text{m}}$)^m]
m	=	material constant
ΔK	=	$K_{max} - K_{min}$ = stress intensity factor range [MPa$\sqrt{\text{m}}$]
K_{max}	=	maximum stress intensity factor in one load cycle [MPa$\sqrt{\text{m}}$]
K_{min}	=	minimum stress intensity factor in one load cycle [MPa$\sqrt{\text{m}}$]

Introducing the step length δa, (3.4) becomes

$$\frac{\delta a}{\mathrm{E}[\delta N]} = \lambda \, C \, (\Delta K)^m \tag{3.5}$$

where

$\mathrm{E}[\delta N]$	=	the expected value of the random variable δN corresponding to the expected numbers of duty cycles which is used to propagate the crack one step δa
λ	=	numbers of load cycles in one duty cycle

Every time the crack tip is exposed to one duty cycle, the same trial is repeated. Thus, the expected numbers of duty cycles is given as the first moment in the geometric distribution, see [Benjamin, J.R. and C.A. Cornell; 1970, p.229], i.e.

$$\mathrm{E}[\delta N] = \sum_{\delta n=1}^{\infty} \delta n \, q(1-q)^{\delta n - 1} = \frac{1}{q} \tag{3.6}$$

Insertion of (3.6) into (3.5) leads to

$$q = \frac{\lambda \, C}{\delta a}(\Delta K)^m \tag{3.7}$$

This means, that for a given material the transition matrix (2.2) is determined. The damage states in structures, made of the given material, are then calculated using (2.5). Account of the geometry of the structure is taken through the stress intensity factor range ΔK.

Generally, ΔK is variable due to the load. Two cases are considered: constant-amplitude loading resulting in slowly varying ΔK and random loading resulting in random ΔK.

4. CONSTANT-AMPLITUDE LOADING

In this chapter, the FM-model for constant-amplitude loading is considered, i.e. the stress range $\Delta\sigma$ is constant. Hence, Paris law (3.5) can be used as crack propagation law. Because $\Delta\sigma$ is constant, the stress intensity factor range ΔK varies slowly, and therefore, ΔK can be assumed constant in the vicinity of the given crack position.

At each crack position, given by (3.1), the random variable δN_j, which is the numbers of duty cycles applied to propagate the crack one step from a_j to $a_j + \delta a$, is considered, see also figure 4.1. It is assumed that a_0 is constant so that the δN_j-values express the properties of the material.

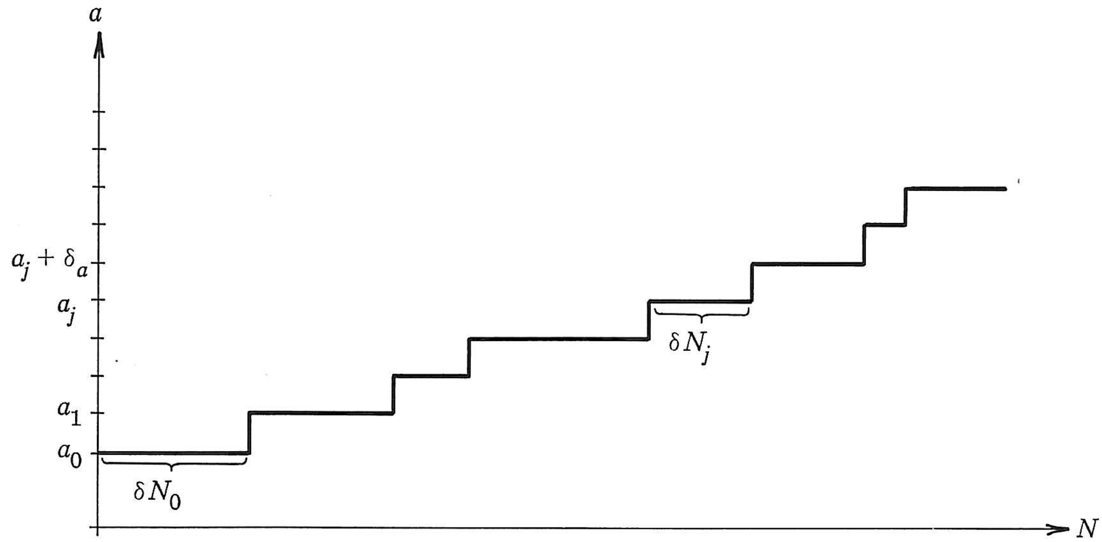

Figure 4.1: Crack propagation in the FM-model. a = crack length, N = numbers of duty cycles.

The numbers of duty cycles performed to propagate the crack δa, is found. The first $(\delta n_j - 1)$ duty cycles with the probability $p_j^{(\delta n_j - 1)}$ do not lead to crack propagation, whereas the δn_j th duty cycle with probability q_j results in crack propagation. Thus, the probability distribution of δN_j is a geometric distribution given as:

$$\mathrm{P}[\delta N_j = \delta n_j] = \mathrm{f}_{\delta N_j}(\delta n_j) = q_j \, p_j^{\delta n_j - 1} \qquad j = 0, 1, 2, \ldots, b-1 \tag{4.1}$$

where

$$q_j = \frac{\lambda C}{\delta a}(\Delta K(a_0 + j\delta a))^m = \frac{\lambda C}{\delta a}(\Delta K(a_j))^m \tag{4.2}$$

$$p_j = 1 - q_j$$

The expected value and the variance of δN_j are given as, see e.g. [Benjamin, J.R. and C.A. Cornell; 1970, p.229-230]:

$$\mathrm{E}[\delta N_j] \; = \; \sum_{\delta n_j=0}^{\infty} \delta n_j \, \mathrm{f}_{\delta N_j} \; = \; \frac{1}{q_j} \tag{4.3}$$

$$\mathrm{Var}[\delta N_j] \; = \; \mathrm{E}[N_j^2] - (\mathrm{E}[N_j])^2 \; = \; \frac{1-q_j}{q_j^2} \tag{4.4}$$

The distribution of δN_j is illustrated in figure 4.2.

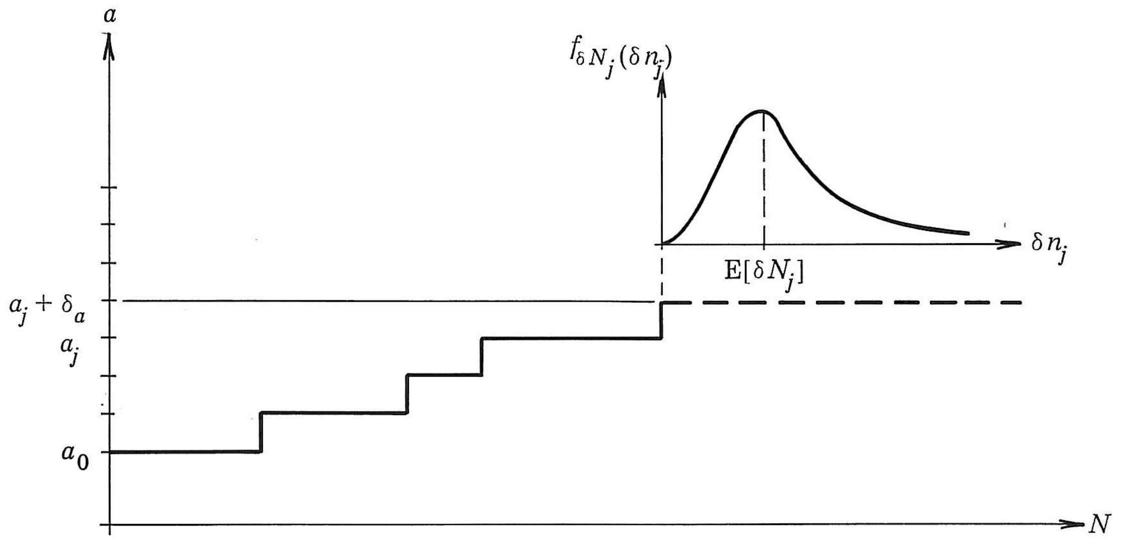

Figure 4.2: The probability distribution of the numbers of duty cycles δN_j performed in state a_j to go to state $a_j + \delta a$.

The total numbers of duty cycles applied to the structure to propagate the crack to the crack length a_j is

$$N \; = \; \sum_{k=0}^{j-1} \delta N_k \tag{4.5}$$

Because the random variable is a sum of independent random variables, the expected value of numbers of duty cycles is

$$\mathrm{E}[N] = \mathrm{E}\left[\sum_{k=0}^{j-1} \delta N_k\right] = \sum_{k=0}^{j-1} \mathrm{E}[\delta N_k] = \sum_{k=0}^{j-1} \frac{1}{q_k} \tag{4.6}$$

Inserting (4.2) and approximating the sum with an integral lead to

$$\mathrm{E}[N] = \frac{\delta a}{\lambda C} \sum_{k=0}^{j-1} (\Delta K_k)^{-m} \; \simeq \; \frac{1}{\lambda C} \int_{a_0}^{a} (\Delta K(a))^{-m} \, da \tag{4.7}$$

which for $\lambda = 1$ corresponds to the integrated Paris law.

The variance of a sum of independent variables is given as, see e.g. [Benjamin, J.R. and C.A. Cornell; 1970, p.227]

$$\mathrm{Var}[N] = \sum_{k=0}^{j-1} \mathrm{Var}[\delta N_k] = \sum_{k=0}^{j-1} \frac{1-q_k}{q_k^2} \tag{4.8}$$

Inserting (4.2) and approximating the sum with an integral lead to

$$\begin{aligned} \mathrm{Var}[N] &= \frac{\delta a}{\lambda^2 C^2} \sum_{k=0}^{j-1} \delta a \left[(\Delta K(a_k))^{-2m} - \frac{\lambda C}{\delta a} (\Delta K(a_k))^{-m} \right] \\ &\simeq \frac{\delta a}{\lambda^2 C^2} \int_{a_0}^{a_j} \left[(\Delta K(a))^{-2m} - \frac{\lambda C}{\delta a} (\Delta K(a))^{-m} \right] \delta a \\ &= \frac{\delta a}{\lambda^2 C^2} f^2(a) - \frac{1}{\lambda C} f(a) \end{aligned} \tag{4.9}$$

where

$$f(a) = \int_{a_0}^{a_j} (\Delta K(a))^{-m}\, \delta a \tag{4.10}$$

The distribution of N is illustrated in figure 4.3.

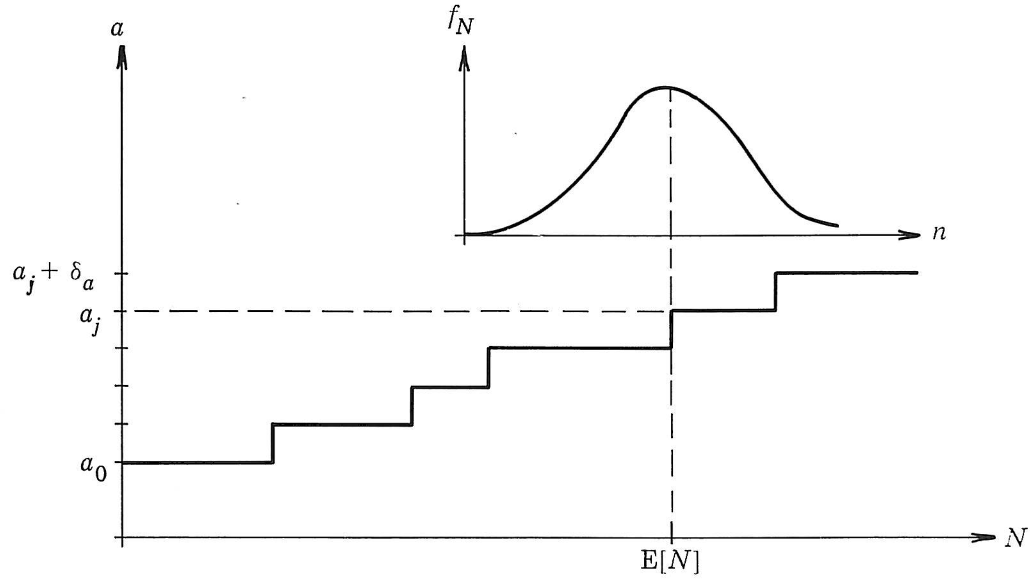

Figure 4.3: The probability distribution of the numbers of duty cycles N performed to go to state a_j.

Notice that, the mean curve corresponds to the values of the material constants C and m.

Considering an infinite plate with a centre crack, the stress intensity factor range is given by

$$\Delta K = \Delta\sigma\sqrt{\pi a} \tag{4.11}$$

where

$$\begin{aligned} \Delta\sigma &= \text{stress range [MPa]} \\ a &= \text{crack length [mm]} \end{aligned}$$

Inserting (4.11) into (4.9) leads to

$$f(a) = \frac{2}{\pi}\frac{(\Delta\sigma)^{-m}}{(m-2)}\left[\frac{1}{(\pi a_0)^{\frac{m}{2}-1}} - \frac{1}{(\pi a)^{\frac{m}{2}-1}}\right] \tag{4.12}$$

which for a (corresponding to $a_{cr} \gg a_0$ where a_{cr} is the critical crack length and a_0 is the initial crack length) approximates

$$f(a) = \frac{2}{\pi}\frac{(\Delta\sigma)^{-m}}{(m-2)}\frac{1}{(\pi a_0)^{\frac{m}{2}-1}} \tag{4.13}$$

Introducing

$$\gamma_0 = \left.\frac{da}{dN}\right|_{a=a_0} = \lambda\, C(\Delta\sigma\sqrt{\pi a_0})^m \tag{4.14}$$

into (4.9) together with (4.13) results in

$$\text{Var}[N] = \left[\frac{2}{(m-2)}\frac{a_0}{\gamma_0}\right]^2 \delta a - \left[\frac{2}{(m-2)}\frac{a_0}{\gamma_0}\right] \tag{4.15}$$

For steel, $m \approx 3$, [Gurney, T.R.; 1979, p.62], thus (4.15) is

$$\text{Var}[N] = 2\frac{a_0}{\gamma_0}\left[2\frac{a_0}{\gamma_0}\,\delta a - 1\right] \tag{4.16}$$

The only unknown quantity left in (4.16) is the step length δa. The value of δa can be estimated if the variance of N is known from experiments.

In [Gurney, T.R.; 1979, p.361] the variance of $ln(N)$ for welds is stated as 0.5. C.f. [Benjamin, J.R. and C.A. Cornell; 1970, p.180],

$$\text{Var}[ln(N)] \approx \left[\left.\frac{\partial(ln(N))}{\partial N}\right|_{\mu_N}\right]^2 \text{Var}[N] \tag{4.17}$$

where

$$\mu_N \; = \; \mathrm{E}[N] \; = \; \text{expected value of the total numbers of duty cycles}$$

(4.17), (4.6) and (4.16) lead to

$$\delta a = \frac{\mathrm{Var}[ln(N)] \left(\sum_{k=0}^{j-1} \frac{1}{q_k} \right)^2 \gamma_0^2 + 2\, a_0\, \gamma_0}{4\, a_0^2} \tag{4.18}$$

in which all quantities are known.

5. RANDOM LOADING

The loading on most structures is of random character, which is more difficult to handle with than deterministic loading.

Due to the randomly varying loading, the stress intensity factor will also vary randomly. The value of the stress intensity factor range, ΔK, depends on the definition of a load cycle.

A realization of a stochastic load process is shown in figure 5.1. Half a stress range is defined as the part of the realization between two adjacent points of reversal, i.e.

$$\frac{1}{2}\,\Delta\sigma \; = \; \sigma_1 - \sigma_2 \tag{5.1}$$

where

$$\begin{array}{lcl} \Delta\sigma & = & \text{stress range [MPa]} \\ \sigma_1 & = & \text{maximum stress [MPa]} \\ \sigma_2 & = & \text{minimum stress [MPa]} \end{array}$$

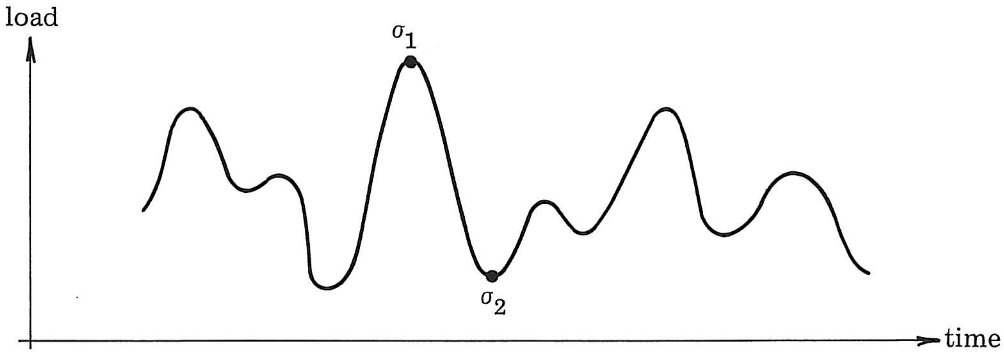

Figure 5.1: Realization of a stochastic load process. σ_1 and σ_2 correspond to the maximum stress and the minimum stress, respectively, in the forthcoming half load cycle.

The FM-model itself does not take into account the well-known effects of acceleration and retardation. This can be done using a crack closure model when ΔK is calculated.

Considering the realization in figure 5.1, the damage increment in the next half cycle will depend on the loading history, the geometry of the structure and the extreme values σ_1 and σ_2. The progress of the effective stress intensity factor, corresponding to the realization in figure 5.1, is shown in figure 5.2.

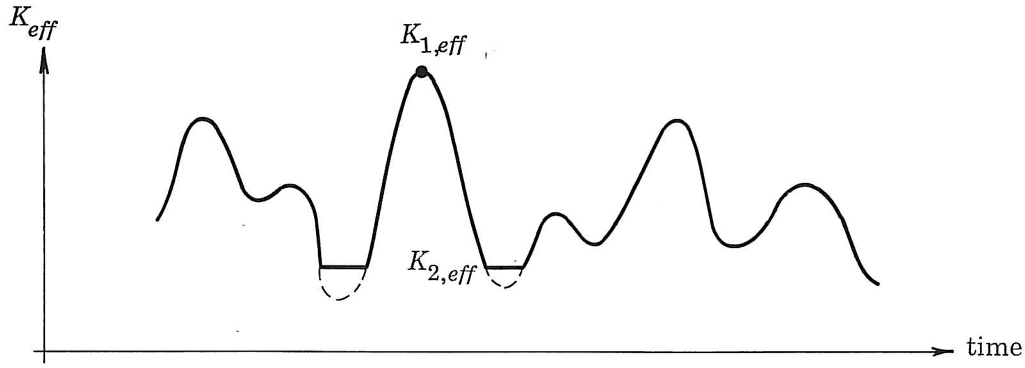

Figure 5.2: Progress of effective stress intensity factor, K_{eff}, corresponding to the realization in figure 5.1. $K_{1,eff}$ and $K_{2,eff}$ correspond to the maximum and the minimum effective stress intensity factor, respectively, in the forthcoming half cycle.

Introducing the effective stress intensity factor range,

$$\Delta K_{eff} = \Delta\sigma_{eff}\, F\, \sqrt{\pi\, a} = (\sigma_{max} - \sigma_{min})\, F\, \sqrt{\pi\, a} \tag{5.2}$$

where

$\Delta\sigma_{eff}$ = effective stress range [MPa]

σ_{max} = maximum stress [MPa]
σ_{cl} = crack closure stress [MPa]
F = geometry function
a = crack length

Paris law (3.4) becomes

$$\frac{da}{dN} = C(\Delta K_{eff})^m \tag{5.3}$$

The transition probability (4.2) changes to

$$q_j = \frac{\lambda C}{\delta a}(\Delta K_{eff}(a_j))^m \tag{5.4}$$

whereas the expected value and variance of the numbers of duty cycles to propagate the crack one step from a_j to $a_j + \delta a$, (4.3) and (4.4) remain

$$\mathrm{E}[\delta N_j] = \frac{1}{q_j} \tag{5.5}$$

$$\mathrm{Var}[\delta N_j] = \frac{1-q_j}{q_j^2} \tag{5.6}$$

The total numbers of duty cycles applied to the structure to propagate the crack to the crack length a_j, (4.5), and the expected value (4.6) are also unchanged:

$$N = \sum_{k=0}^{j-1} \delta N_k \tag{5.7}$$

$$\mathrm{E}[N] = \sum_{k=0}^{j-1} \frac{1}{q_k} \tag{5.8}$$

The approximate value of (5.8) becomes

$$\mathrm{E}[N] \simeq \frac{1}{\lambda C}\int_{a_0}^{a} (\Delta K_{eff}(a))^{-m}\, da \tag{5.9}$$

The variance (4.8) is still given as

$$\mathrm{Var}[N] = \sum_{k=0}^{j-1} \frac{1-q_k}{q_k^2} \tag{5.10}$$

with (4.9) as the approximative value,

$$\mathrm{Var}[N] \simeq \frac{\delta a}{\lambda^2 C^2} f^2(a) - \frac{1}{\lambda C} f(a) \tag{5.11}$$

but, where

$$f(a) = \int_{a_0}^{a_j} (\Delta K_{eff}(a))^{-m} \, \delta a \tag{5.12}$$

Thus, the FM-model is also available if the loading is random.

6. CONCLUSIONS

On the basis of the well-known B-model, see chapter 2, a new numerical cumulative damage model based on fracture mechanics is introduced. This model, the FM-model described in chapter 3-5, incorporates existing physical knowledge of crack propagation.

The cumulative damage is described by a discrete-time, discrete-state Markov process. The time is measured in numbers of duty cycles, whereas the state of damage is given as a crack length. The crack is assumed to propagate in steps of the length δa.

The Markov assumption is reasonable for small values of the step length, δa. Further, Paris law is used as crack propagation law.

One of the main results is that the step length δa appears to be a material constant so that, once the materials constants in Paris law, C and m, and δa are determined, the state of damage in any structure of the given material can be calculated numerically.

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SYMBOLS

a	=	crack length
a_0	=	initial crack length
a_b	=	failure crack length
$\delta a, \Delta a$	=	crack length increment
b	=	failure state
C	=	material constant
CD	=	cumulative damage
d	=	damage state
da	=	crack length increment
DC	=	duty cycle
$dN, \Delta N$	=	increase in numbers of cycles
f_X	=	probability mass function
K	=	stress intensity factor
K_{max}	=	maximum stress intensity factor
K_{min}	=	minimum stress intensity factor
ΔK	=	stress intensity factor range
ΔK_{eff}	=	effective stress intensity factor range
m	=	material constant
N	=	numbers of cycles
δN	=	increase in numbers of duty cycles
$\overline{p}_0$	=	initial state vector
p	=	$1 - q$
$\overline{p}_x$	=	state vector
$\overline{\overline{P}}$	=	transition matrix
q	=	transition probability
x	=	time
X	=	random variable
γ_0	=	initial crack growth rate
λ	=	numbers of load cycles in one duty cycle
μ	=	mean value
π_j	=	probability that damage is initially in state j
$\Delta\sigma$	=	stress range
$\Delta\sigma_{eff}$	=	effective stress range
σ_{cl}	=	crack closure stress

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